Divergence and curl

Tuesday, August 18, 2020 3:58 PM

vector field describe the flow of some substance (electricity, wind or fluid)

Divergence & curl are the characteristics of how flow is behaving on a vector fields in a small neighbourhood around a given point P.

<u>Divergence</u>: A measurement of how much fluid/flow enters the neighbourhood around P compared to how much leaves.

→ If more fluid/flow enters the neighbourhood than leaves it, than divergence will be (−) at 'P'

* This fluid/flow gathering at the point.

→ If the same amount of fluid/flow enters as leaves, then divergence will be '0' at P.

<u>Incompressive</u>.

Incompressible flow implies that the density remains constant within a parcel of fluid that moves with the flow velocity.

→ If more fluid leaves than enters then divergence will be (+) at 'P'.

$$F(x,y,z) = (P)\hat{i} + Q\hat{j} + R\hat{k}$$

(del) $\nabla = \left(\frac{\partial}{\partial x}\right)\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$

$$\text{Div } F = \nabla \cdot F = \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right)$$

$$\text{Div } F = \nabla \cdot F = \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right)$$

Scalar

Curl: The measurement of rotation of the vector field in the neighborhood around 'P'.

How much the paddle spins.

→ If curl (+) at a point then flow/fluid paddle would rotate (counterclockwise)

→ If curl (−) – Rotation clockwise at 'P'

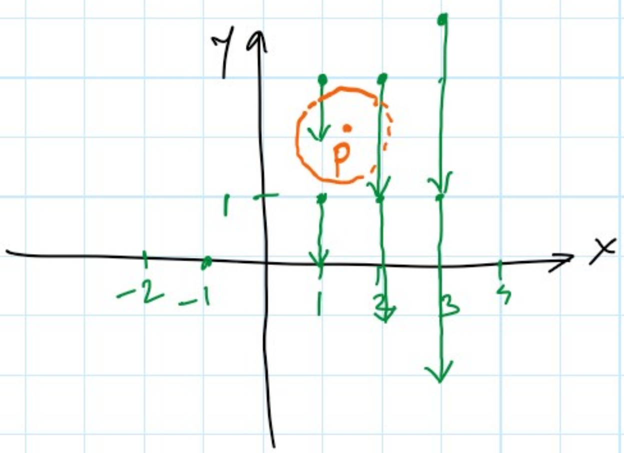
→ If curl = 0 → No rotation at 'P'

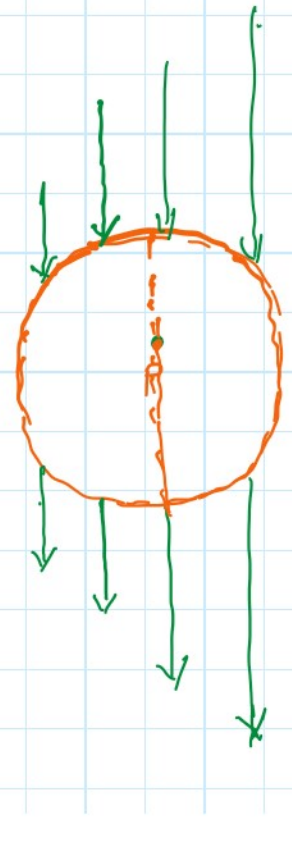
Irrotational.

$$\text{Curl } F = \nabla \times F$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \left(\frac{\partial}{\partial y}(R)\right.$$

Ex $F(x, y, z) = -x\hat{j}$





Div F = 0

curl F = −ve

$$\text{Div}F = \nabla\cdot F = \frac{\partial}{\partial x}(P) + \frac{\partial}{\partial y}(Q) + \frac{\partial}{\partial z}(R)$$

$$= \frac{\partial}{\partial x}(0) + \frac{\partial}{\partial y}(-x) + \frac{\partial}{\partial z}(0)$$

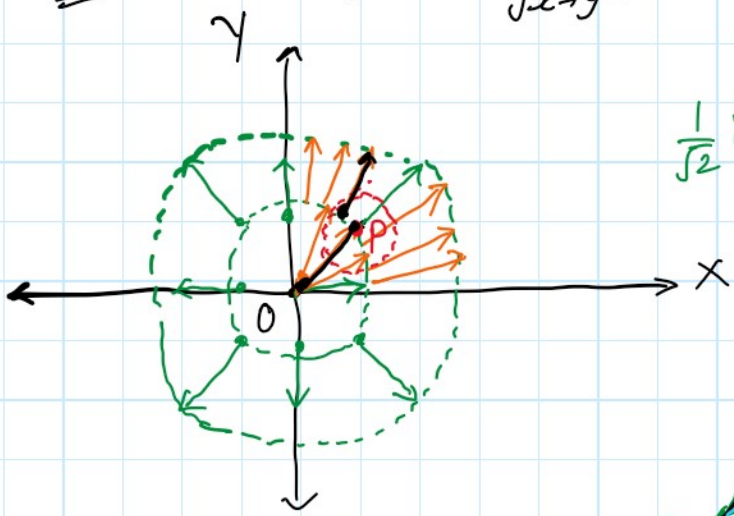
$$= 0 + 0 + 0 = 0$$

$$\text{curl } F = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & -x & 0 \end{vmatrix} = (0-0)\hat{i} - (0-0)\hat{j} + (-1-0)\hat{k}$$

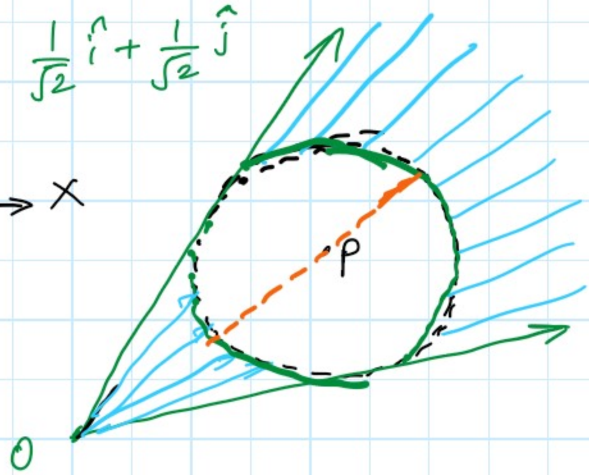
$$= -\hat{k}$$

$$\text{curl } F = -\hat{k}$$

Ex $\quad F(x,y,z) = \frac{x}{\sqrt{x^2+y^2}}\hat{i} + \frac{y}{\sqrt{x^2+y^2}}\hat{j}$



$\frac{1}{\sqrt{2}}\hat{i} + \frac{1}{\sqrt{2}}\hat{j}$



Div F = +ve.

curl F = 0

$$\text{Div } F = \nabla\cdot F = \frac{\partial}{\partial x}\left(\frac{x}{\sqrt{x^2+y^2}}\right) + \frac{\partial}{\partial y}\left(\frac{y}{\sqrt{x^2+y^2}}\right) + \frac{\partial}{\partial z}(0)$$

$$= \frac{\sqrt{x^2+y^2} - x \times \frac{1}{2\sqrt{x^2+y^2}}\cdot 2x}{(x^2+y^2)} + \frac{\sqrt{x^2+y^2} - \frac{y\cdot 2y}{2\sqrt{x^2+y^2}}}{(x^2+y^2)}$$

$$\cancel{x^2}+y^2-\cancel{x^2} \qquad x^2+\cancel{y^2}-\cancel{y^2}$$

$$= \frac{\cancel{x^2}+y^2-\cancel{x^2}}{(x^2+y^2)\sqrt{x^2+y^2}} + \frac{x^2+\cancel{y^2}-\cancel{y^2}}{(x^2+y^2)\sqrt{x^2+y^2}}$$

$$\text{Div } F = \frac{1}{\sqrt{x^2+y^2}} > 0$$

$$\text{Curl } F = \nabla \times F = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{x}{\sqrt{x^2+y^2}} & \frac{y}{\sqrt{x^2+y^2}} & 0 \end{vmatrix}$$

$$= (0-0)\hat{i} - (0-0)\hat{j} + \left[-\frac{1}{2}(2x)\,y\,(x^2+y)^{-3/2} - \left(-\frac{1}{2}\right)(2y)\,x\,(x^2+y^2)^{-3/2}\right]\hat{k}$$

$$= 0$$

Ex :- $F(x,y,z) = yz\,\hat{i} + xz\,\hat{j} + xy\,\hat{k}$

$$\text{Div } F = \nabla \cdot F = \frac{\partial}{\partial x}(yz) + \frac{\partial}{\partial y}(xz) + \frac{\partial}{\partial z}(xy)$$

$$= 0$$

Incompressive.

$$\text{Curl } F = \nabla \times F = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & xz & xy \end{vmatrix}$$

$$= (x-x)\hat{i} - (y-y)\hat{j} + (z-z)\hat{k} = 0$$

Irrotational.

Ex find Div F & curl F for

$$F(x,y,z) = xy\,\hat{i} + xz\,\hat{j} + xyz^2\,\hat{k}$$

at $P(-1,2,1)$

Sol^n:

$$\text{Div} F = \nabla \cdot F = \frac{\partial}{\partial x}[xy] + \frac{\partial}{\partial y}[xz] + \frac{\partial}{\partial z}[xyz^2]$$

$$= y + 0 + 2xyz$$

$$\text{Div} F = y + 2xyz$$

$$\text{Div} F_{(-1,2,1)} = 2 + 2(-1)(2)(-1)$$
$$= -2$$

$$\text{curl} F = \nabla \times F = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & xz & xyz^2 \end{vmatrix}$$

$$= (xz^2 - x)\,\hat{i} - (yz^2 - 0)\,\hat{j} + (z - x)\,\hat{k}$$

$$\text{Curl } F = (-1+1)\,\hat{i} - (2-0)\,\hat{j} + (1+1)\,\hat{k}$$

$$\text{curl} F = -2\hat{j} + 2\hat{k}$$

$$\text{Max rotation} = \|\text{curl } F\|$$

$$= \sqrt{8} = 2\sqrt{2}.$$

Max rotation happens only if the paddle $\parallel$ to curl F.